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# Penrose limit of $AdS_3 \times S^3 \times S^3 \times S^1$ and its associated $\sigma$-model

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Abstract: We study the Penrose limit of a supersymmetric IIB background with non-trivial NS 3-form field strength, obtaining a solution with the smallest number of supercharges (i.e. 16) allowed; we write down explicitly the superalgebra of the theory, build the supersymmetric associated IIB string $\sigma$-model and make conjectures on the dual gauge theory.

Keywords: Sigma Models, Superstrings and Heterotic Strings, Penrose limit and pp-wave background.



 

## Contents





## 1. Introduction

Since the middle 70's, it is believed that large $N$ gauge theories have a string theory description, even if these strings live in a higher dimensional space than gauge theory; this idea stands as the basis of the so-called $AdS/CFT$ correspondence (for an exhaustive review, see [1]), where a conformal field theory (CFT) in $(p+1)$ dimensions is related to superstring theory in $AdS_{p+2} \times X$, with $X$ a generic Einstein compact manifold.

Unfortunately, it is not possible to study the complete spectrum of the string theory in any arbitrary background, due to technical difficulties in quantization, and one is obliged to study it only for large values of the radius of both $AdS$ and $X$ , that is in a supergravity approximation; but, being $R \sim \left(g^2_{\mathrm{YM}} N\right)^1 /4$, this means that only the sector of the SCFT with large 't Hooft coupling is involved.

Going beyond this limit is essential, for instance, in order to find quantitative results from string theory for the large $N$ limit of gauge theories with finite 't Hooft coupling and also to prove (and not only to conjecture) the $AdS/CFT$ correspondence.

But actually AdS is not the simplest supergravity background: Blau, Figueroa-O'Farrill, Hull and Papadopoulos [2] demonstrated that there exists another maximally supersymmetric background in IIB supergravity, which can be obtained as the Penrose limit of the $AdS_5 \times S_5$ background, and whose metric is non-flat only in two directions:

$$ds^2 = 2dudv + \sum_{A,B=1}^{8} \left(\mathcal{A}^{AB}\ y_A y_B\right) du^2 + \sum_{A=1}^{8} (dy_A)^2 \tag{1.1}$$

with a non-trivial 5-form:

$$F_{u1234} = F_{u5678} = \text{const.} \tag{1.2}$$

Subsequently, it was shown [3, 4] that these types of background lead to a superstring $\sigma$-model that is easily quantized in the light-cone gauge, since it is quadratic in all the fields. The next important step in this direction was achieved by Berenstein, Maldacena and Nastase (BMN) [5], who extended the $AdS/CFT$ duality to pp-wave background in string theory, by claiming that the string theory in a background obtained from $AdS_5 \times S_5$ after a Penrose limit should be dual to the large $N$ limit of a certain sub-sector of four dimensional $\mathcal{N}$=4 SYM.

This sub-sector is characterized by choosing a $\mathrm{U}(1)_R$ subgroup of the usual $\mathrm{SU}(4)_R$ R-symmetry group of the gauge theory and by considering states with conformal weight $\Delta$ and R-charge $J$ scaling as $\sqrt{N}$, so that both of them are large, but with $\Delta - J$ finite.

After [5], the Penrose limit was applied to other models, with different amount of supersymmetries and different geometries, and some results were obtained about the non-perturbative behaviour of a large variety of gauge theories; in particular, in [6] superstrings on the Penrose limits of $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times K^3$ have been investigated; in [7], BMN duality is tested when the superstring background is $AdS_3 \times S_3 \times K^3$, finding an agreement between CFT particle and string spectra at least in the leading order in $J^2/N$, while in [8] a map between CFT operators and string oscillators is found in an $AdS_3 \times S^3 \times T^4$ background; in [9], the Penrose limit of $AdS_3 \times S^3 \times T^4$ is studied together with its supersymmetry content, and in [10] solutions of the type $AdS_3 \times S_3$ are studied with either NS-NS or R-R 3-form background.

Motivated by these considerations, and moving from our precedent work where we studied compactifications of IIB supergravity on generic backgrounds of the type $AdS_3 \times G/H$ [11], in this paper we study the Penrose limit of a IIB configuration with non-trivial NS 3-form, its supersymmetry content and the supersymmetric $\sigma$-model hamiltonian in this background; the paper is organized as follows: in section 2 and 3 we perform the Penrose limit of our supergravity background and calculate the spin connection, the Riemann and the Ricci tensors. In section 4, we study the supersymmetry variations of gravitino and dilatino, and show that there is no supersymmetry enhancement; moreover, the number of supersymmetry charges conserved is the minimum for a pp-wave background, and this leads to the fact that bosons and fermions in the $\sigma$- model have different masses (that is, the supersymmetry is completely broken in the light-cone gauge [12]).

In section 5 we calculate explicitly the supersymmetry algebra: the bosonic sector is obtained as an Inönü-Wigner contraction of the parent algebra, while the mixed and fermionic sectors are obtained by means of the spinorial Lie derivative and of the structure of the Killing spinors.

In section 6 we write down the supersymmetric $\sigma$-model lagrangian both in bosonic and in fermionic sectors, and we can observe another interesting feature of this model: as a remnant of the parent background, where the radii of $AdS$ and $S$ were different, the eight bosons are divided into three sets: four of them are massless, coming from the $S^3 \times S^1$ part trivially involved in the limit, two have mass $\frac{1}{2}\mu^2$ and the others $\frac{1}{4}\mu^2$, being respectively the embedding fields coming from coordinates of $AdS_3$ and $S^3$.

In section 7, we compute explicitly the supersymmetric $\sigma$-model hamiltonian and evaluate its spectrum, which is in perfect agreement with that found by BMN, where:

$$\begin{aligned} H_{\text{string}} &= \omega_n \mathcal{N}_n \\ \omega_n &= \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} \end{aligned} \tag{1.3}$$

with the only exception for an interaction term that generates an extra piece in the eigenvalues of the string hamiltonian:

$$\omega_n = \sqrt{a\,(\mu p^+ \alpha')^2 \pm b\,(\mu p^+ \alpha')\,n + n^2}\,, \tag{1.4}$$

where $a$ and $b$ are constants.

Finally, section 8 contains some comments about the SCFT dual to the parent background and about BMN duality in this case.

## 2. Penrose limit of $AdS_3 \times S_3 \times S_3 \times S_1$

Let us start with a particular solution of type IIB supergravity, $AdS_3 \times S_3 \times S_3 \times S_1$ [11], with nonzero 3-form $G$, whose metric reads:

$$\begin{aligned} ds^2 = {}& R^2[\cosh^2\rho\; dt^2 - d\rho^2 - \sinh^2\rho\; d\alpha^2 - 2(\cos^2\theta_1\; d\psi_1^2 + d\theta_1^2 + \sin^2\theta_1\; d\phi_1^2 + \\ & + \cos^2\theta_2\; d\psi_2^2 + d\theta_2^2 + \sin^2\theta_2\; d\phi_2^2 + d\theta^2)]\,, \end{aligned} \tag{2.1}$$

where $\rho$, $\alpha$ and $t$ are global coordinates (see appendix B) on $AdS_3$, whose radius is $R$; $\theta_{1,2}$, $\psi_{1,2}$ and $\phi_{1,2}$ are coordinates of the two 3-spheres, whose radius is $\sqrt{2}R$, and $\theta$ parametrizes $S_1$; finally, $G$ is defined by:

$$\begin{aligned} G = {}& 2\sqrt{2}R^2\cosh\rho\;\sinh\rho\; dt\; d\rho\; d\alpha + 4\sqrt{2}R^2\sin\theta_1\;\cos\theta_1\; d\psi_1\; d\theta_1\; d\phi_1 + \\ & + 4\sqrt{2}R^2\sin\theta_2\;\cos\theta_2\; d\psi_2\; d\theta_2\; d\phi_2\,. \end{aligned} \tag{2.2}$$

Notice that, in the light of BMN correspondence, it is convenient to use global coordinates because in this way the space-time energy of a string state is directly related to the conformal dimension of the corresponding CFT state in dual gauge theory. In order to study the pp-wave configuration, we need to change coordinates from (2.1) to:

$$
\begin{aligned}
t &= \frac{\mu}{\sqrt{2}}u + \frac{\omega^2}{\sqrt{2}\mu R^2}v & \theta_1 &= \frac{\omega}{\sqrt{2}R}y_{34} \\
\rho &= \frac{\omega}{R}y_{12} & \psi_2 &= \frac{\omega}{\sqrt{2}R}y_7 \\
\alpha &= \arctan\frac{y_2}{y_1} & \phi_2 &= \arctan\frac{y_6}{y_5} \\
\psi_1 &= \frac{\mu}{2}u - \frac{\omega^2}{2\mu R^2}v & \theta_2 &= \frac{\omega}{\sqrt{2}R}y_{56} \\
\phi_1 &= \arctan\frac{y_4}{y_3} & \theta &= \frac{\omega}{\sqrt{2}R}y_8\,,
\end{aligned}
\tag{2.3}
$$

where $y_{AB} = ({y_A}^2 + {y_B}^2)^{1/2}$ ($A$,$B$ run over $1,\dots,8$), $\omega$ is a rescaling dimensionless parameter and $\mu$ a dimensional parameter which will be interpreted as mass in the light-cone gauge $\sigma$-model lagrangian; these coordinates are well suited to describe the geometry before taking the limit, since they describe the space from the viewpoint of the null geodesic ($\omega$ represents the distance from the null geodesic in units of $R$). Using these definitions in (2.1), we rewrite the metric as function of the new variables:

$$
\begin{aligned}
\frac{ds^2}{R^2} = {} & \frac{\mu^2}{2}\left(\cosh^2\frac{\omega}{R}y_{12} - \cos^2\frac{\omega}{\sqrt{2}R}y_{34}\right)du^2 + \frac{\omega^4}{2\mu^2 R^4}\left(\cosh^2\frac{\omega}{R}y_{12} + \cos^2\frac{\omega}{\sqrt{2}R}y_{34}\right)dv^2 + \\
& + \frac{\omega^2}{R^2}\left(\cosh^2\frac{\omega}{R}y_{12} + \cos^2\frac{\omega}{\sqrt{2}R}y_{34}\right)du\,dv - \left(\frac{\omega^2}{R^2}\frac{{y_1}^2}{y_{12}^2} + \frac{{y_2}^2}{y_{12}^4}\sinh^2\frac{\omega}{R}y_{12}\right){dy_1}^2 - \\
& - \left(\frac{\omega^2}{R^2}\frac{{y_2}^2}{y_{12}^2} + \frac{{y_1}^2}{y_{12}^4}\sinh^2\frac{\omega}{R}y_{12}\right){dy_2}^2 - 2\frac{y_1 y_2}{y_{12}^4}\left(\frac{\omega^2}{R^2}y_{12}^2 - \sinh^2\frac{\omega}{R}y_{12}\right)dy_1 dy_2 - \\
& - \left(\frac{\omega^2}{R^2}\frac{{y_3}^2}{y_{34}^2} + 2\frac{{y_4}^2}{y_{34}^4}\sin^2\frac{\omega}{\sqrt{2}R}y_{34}\right){dy_3}^2 - \left(\frac{\omega^2}{R^2}\frac{{y_4}^2}{y_{34}^2} + 2\frac{{y_3}^2}{y_{34}^4}\sin^2\frac{\omega}{\sqrt{2}R}y_{34}\right){dy_4}^2 - \\
& - 2\frac{y_3 y_4}{y_{34}^4}\left(\frac{\omega^2}{R^2}y_{34}^2 - 2\sin^2\frac{\omega}{\sqrt{2}R}y_{34}\right)dy_3 dy_4 - \left(\frac{\omega^2}{R^2}\frac{{y_5}^2}{y_{56}^2} + 2\frac{{y_6}^2}{y_{56}^4}\sin^2\frac{\omega}{\sqrt{2}R}y_{56}\right){dy_5}^2 - \\
& - \left(\frac{\omega^2}{R^2}\frac{{y_6}^2}{y_{56}^2} + 2\frac{{y_5}^2}{y_{56}^4}\sin^2\frac{\omega}{\sqrt{2}R}y_{56}\right){dy_6}^2 - 2\frac{y_5 y_6}{y_{56}^4}\left(\frac{\omega^2}{R^2}y_{56}^2 - 2\sin^2\frac{\omega}{\sqrt{2}R}y_{56}\right)dy_5 dy_6 - \\
& - \frac{\omega^2}{R^2}\cos^2\frac{\omega}{\sqrt{2}R}y_{56}\,{dy_7}^2 - \frac{\omega^2}{R^2}\,{dy_8}^2\,.
\end{aligned}
\tag{2.4}
$$

Using now the definition of Penrose limit for the metric, i.e. $ds^2_\omega = \lim_{\omega\to 0}\frac{1}{\omega^2}ds^2(\omega)$, we get:

$$
ds^2_\omega = \frac{1}{2}\mu^2\left(y_{12} + \frac{1}{2}y_{34}\right)du^2 + 2du\,dv - dy_A dy_A\,. \tag{2.5}
$$

We could have expected a similar result because geometrically speaking the Penrose limit enlarges and stretches a portion of the parent space near a null geodesic (in this case



JHEP07(2003)035

identified by $\rho=\psi_2=\theta_1=\theta_2=\theta=0$ with affine parameter $u$), the mathematical effect being to unwrap the angular coordinates orthogonally to the $u-v$ plane, making the sphere geometry flat.

The same procedure must be performed [13] over the 3-form (2.2), that is the only nonzero field; after the change of coordinates, we get:

$$\begin{aligned}
G_{u12} &= 2\mu\omega R\,\frac{1}{y_{12}}\,\cosh\frac{\omega}{R}y_{12}\,\sinh\frac{\omega}{R}y_{12}\\
G_{v12} &= \frac{2\omega^3}{\mu R}\,\frac{1}{y_{12}}\,\cosh\frac{\omega}{R}y_{12}\,\sinh\frac{\omega}{R}y_{12}\\
G_{u34} &= 2\mu\omega R\,\frac{1}{y_{34}}\,\sin\frac{\omega}{\sqrt{2}R}y_{34}\,\cos\frac{\omega}{\sqrt{2}R}y_{34}\\
G_{v34} &= \frac{2\omega^3}{\mu R}\,\frac{1}{y_{34}}\,\sin\frac{\omega}{\sqrt{2}R}y_{34}\,\cos\frac{\omega}{\sqrt{2}R}y_{34}\\
G_{567} &= -2\sqrt{2}\omega^2\,\frac{1}{y_{56}}\,\sin\frac{\omega}{\sqrt{2}R}y_{56}\,\cos\frac{\omega}{\sqrt{2}R}y_{56}
\end{aligned} \tag{2.6}$$

and, since $G^\omega=\lim_{\omega\to 0}\frac{1}{\omega^2}G(\omega)$,

$$\begin{aligned}
G^\omega_{u12} &= 2\mu\\
G^\omega_{u34} &= \sqrt{2}\mu
\end{aligned} \tag{2.7}$$

while all other components vanish. It is important to notice that this background satisfies the equations of motion of IIB supergravity because of the so-called hereditary property of the Penrose limit [15]. Another remark is in order: the radius of $S^1$ is not really fixed by the equations of motion: so that in (2.5) we have chosen it to coincide with the last $S^3$ radius even if it is not necessary; this fact will become evident later, when we will study the superalgebra of the background.

## 3. Geometry of pp-wave solution

In studying the geometry of configuration (2.1), it is convenient to use flat space light-cone indices (that is $E^\pm=\frac{1}{2}\left(E^0\pm E^9\right)$), the flat metric being (again, $A,B:1,\ldots,8$):

$$\eta_{++}=\eta_{--}=0\,;\qquad \eta_{+-}=1\,;\qquad \eta_{AB}=-\delta_{AB} \tag{3.1}$$

the vielbein ($E^M_\mu$) of our background and its inverse is:

$$\begin{aligned}
E^+_u &= 1 & E^u_+ &= 1\\
E^+_v &= 0 & E^v_+ &= -\frac{1}{4}\mu^2\left(y_{12}^2+\frac{1}{2}y_{34}^2\right)\\
E^-_u &= \frac{1}{4}\mu^2\left(y_{12}^2+\frac{1}{2}y_{34}^2\right) & E^u_- &= 0\\
E^-_v &= 1 & E^v_- &= 1\\
E^A_A &= 1 & E^A_A &= 1\,.
\end{aligned} \tag{3.2}$$

As a consequence, in flat indices, $G$ takes the values:

$$\begin{aligned} G^{\omega}_{+12} &= 2\mu \\ G^{\omega}_{+34} &= \sqrt{2}\mu\,. \end{aligned} \tag{3.3}$$

By means of Cartan-Maurer equations implemented with null-torsion condition ($dE^M + \omega^M_{\ N} \wedge E^N = 0$) , we can write down the riemannian connection:

$$\begin{aligned} \omega_{u\ +i} &= \frac{1}{2}\mu^2 y_i \\ \omega_{u\ +m} &= \frac{1}{4}\mu^2 y_m \end{aligned} \tag{3.4}$$

and subsequently the Riemann tensor, defined as $\mathcal{R}_{MN} = d\omega_{MN} + \omega_M^{\ \ P} \wedge \omega_{PN}$; finally, we obtain Ricci tensor ($R_{MN} = \mathcal{R}^P_{\ \ MPN}$), whose only non zero component is:

$$R_{++} = \frac{3}{2}\mu^2\,, \tag{3.5}$$

from which we can conclude that the curvature scalar ($R = R_{MN}\eta^{MN}$) is zero, being $\eta^{++} = 0$.

## 4. Supersymmetry

It is well known that any pp-wave configuration preserves, at least, all the supersymmetries of its parent background, as shown in [15]; this means that, since our parent background configuration ($AdS_3 \times S_3 \times S_3 \times S_1$ with nonzero 3-form $G$) preserves one half of supersymmetry charges, the pp-wave configuration will preserve at least this same amount. Requiring the vanishing of the dilatino supersymmetry variation brings us to the equation (from now on, we drop the label $\omega$ on the fields):

$$G_{MNP}\Gamma^{MNP}\epsilon = 0 \tag{4.1}$$

after some Dirac algebra, (4.1) can be translated into the condition:

$$\Gamma^+\epsilon = \frac{1}{2}\left(\Gamma^0 + \Gamma^9\right)\epsilon = 0 \tag{4.2}$$

which precisely halves the amount of supersymmetry charges preserved: we can guess now that imposing the vanishing of the gravitino variation will not reduce any further the total amount of supersymmetry, but only determine the exact dependence of $\epsilon$ on the coordinates; the explicit expression for $\delta\psi_M = 0$ is:

$$\nabla_M\epsilon + \frac{1}{96}\left(\Gamma_M^{\ \ NPQ} G_{NPQ} - 9\Gamma^{PQ} G_{MPQ}\right)\epsilon^* = 0\,, \tag{4.3}$$

where $\nabla_M$ is the standard covariant derivative ($\nabla_M = \partial_M + \frac{1}{4}\omega_{M\ NP}\Gamma^{NP}$). In ten dimensions, spinors satisfy both Majorana and Weyl conditions: Weyl condition says that $\epsilon$ has a definite chirality, and then only one half of its components are nonzero, while as



JHEP07(2003)035

a consequence of Majorana condition, we can easily separate the real from the imaginary part of equation (4.3); besides, after some algebra, we can rewrite the gravitino variation as follows:

$$\nabla_M \epsilon^I + \sigma_3^{IJ} \frac{1}{96} \left( \Gamma_M \Gamma^{NPQ} G_{NPQ} - 12 \Gamma^{PQ} G_{MPQ} \right) \epsilon^J = 0 \,, \tag{4.4}$$

where $I, J = 1, 2$ label the real and imaginary part of $\epsilon$ (that is $\epsilon = \epsilon^1 + i\epsilon^2$), and $\sigma_3$ is the usual third Pauli matrix; the first supersymmetry condition (4.2) simplifies equation (4.3), and we get:

$$\left( \delta^{IJ} \nabla_M - \sigma_3^{IJ} \frac{1}{8} G_{MNP} \Gamma^{NP} \right) \epsilon^J = 0 \,. \tag{4.5}$$

Due to our background, in the directions from 5 up to 8, this condition simply reads:

$$\partial_q \epsilon^I = 0 \,. \tag{4.6}$$

Now we focus on $i$ and $m$ directions, where the riemannian connection is always zero, while the $G$-term, after some manipulations, takes the form:

$$\begin{aligned} G_{iMN} \, \Gamma^{MN} &= 2\mu \Gamma_i W \Gamma^+ \\ G_{mMN} \, \Gamma^{MN} &= \sqrt{2} \mu \Gamma_m Y \Gamma^+ \end{aligned} \tag{4.7}$$

and the matrices $W$ and $Y$ are defined as:

$$W = \frac{1}{2} \varepsilon_{ij} \, \Gamma^{ij} = \Gamma^1 \Gamma^2 \,, \qquad Y = \frac{1}{2} \varepsilon_{mn} \, \Gamma^{mn} = \Gamma^3 \Gamma^4 \,, \tag{4.8}$$

recalling condition (4.2), we can rewrite equation (4.5) as:

$$\partial_i \epsilon^I = \partial_m \epsilon^I = 0 \,. \tag{4.9}$$

In the $-$ and $+$ directions, instead, we get:

$$\partial_- \epsilon^I = 0 \,, \qquad \partial_+ \epsilon^I = \partial_u \epsilon^I = -\sigma_3^{IJ} \frac{1}{4} \mu \left( 2W + \sqrt{2} Y \right) \epsilon^J \tag{4.10}$$

differential equations (4.6),(4.9) and (4.10) tell us that $\epsilon$ is independent on $y_q$, $y_i$, $y_m$and $v$, while retaining the typical non trivial dependence on $u$, in fact the second of (4.10) can be easily integrated, being a linear first order differential equation with constant coefficients, and the explicit solution turns out to be:

$$\begin{aligned} \epsilon(u)^I &= \left[ -\sigma_3^{IJ} \sin\left( \frac{1}{2} \mu u \right) W + \cos\left( \frac{1}{2} \mu u \right) \mathbb{1} \delta^{IJ} \right] \times \\ &\quad \times \left[ -\sigma_3^{IJ} \sin\left( \frac{\sqrt{2}}{4} \mu u \right) Y + \cos\left( \frac{\sqrt{2}}{4} \mu u \right) \mathbb{1} \delta^{IJ} \right] \epsilon(0)^J \,, \end{aligned} \tag{4.11}$$

where $\epsilon(0)^I$ satisfies $\Gamma^+ \epsilon(0)^I = 0$; there is no supersymmetry enhancement: only one half of supersymmetry charges are conserved.

It is important to notice that, with a different ansatz, we could preserve a larger amount of supersymmetry: for instance in [14] a rotation of the two foliating circles within the two 3-spheres is taken, and supernumerary Killing spinors arise, when the rotation angle is $\alpha = \pi/4$.



JHEP07(2003)035

## 5. pp-wave solution superalgebra

Any superalgebra $\mathfrak{G}$ can be divided into two subsets: a bosonic one, which we will refer to as $\mathfrak{G}_0$, and a fermionic one, $\mathfrak{G}_1$, spanned respectively by Killing vectors and spinors; both $\mathfrak{G}_0$ and $\mathfrak{G}_1$ will be evaluated starting from the generators of the isometry algebra of the parent background, following the method used in [15].

In order to evaluate the even subalgebra, it is convenient to make a distinction, in the parent background, between the $AdS_3 \times S_3$ part, that is non-trivially modified by the Penrose limit, and $S_3 \times S_1$, where there is no coordinate mixing. The former subspace is invariant under the action of the algebra $\mathfrak{so}(2,2) \oplus \mathfrak{so}(4)$, which can be thought of as the algebra of rotations in the 8-dimensional space $\mathbb{R}^{2,2} \times \mathbb{R}^4$ where $AdS_3$ and $S_3$ are hyper-surfaces (see appendix B) [16]; analogously the $S_3 \times S_1$ subspace is invariant under the algebra $\mathfrak{so}(4) \oplus \mathfrak{so}(2)$ of rotations in the 6-dimensional space $\mathbb{R}^4 \times \mathbb{R}^2$ where again $S_3$ and $S_1$ are hyper-surfaces.

In summary, the parent background can be viewed as a 10-dimensional hyper-surface lying in the space $\mathcal{M} = \mathbb{R}^{2,2} \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^2$, whose generators of isometries are:

$$\xi_{\Lambda\Sigma} = z_\Lambda \partial_\Sigma - z_\Sigma \partial_\Lambda\,, \tag{5.1}$$

where $\partial_\Lambda = \frac{d}{dz^\Lambda}$ and $z_\Lambda$, are cartesian coordinates of any of the four subspaces building blocks of the 14-dimensional $\mathcal{M}$ space.

Actually, to describe the parent space geometry, we adopt global coordinates which are written in terms of the $z_\Lambda$'s in the appendix B. Now (2.3) is to be performed, and the generators turn out to be a complicated mixture of derivative operators multiplied by (hyperbolic) trigonometric functions of $u$ and $v$; before taking the limit, we must rescale the $\xi$'s by an appropriate power of $\omega$, which will be chosen so that all the limits exist and are non-zero:

$$\xi^\omega = \lim_{\omega\to 0} \omega^{\Delta_\xi}\xi,\ \ \Delta_\xi \in \mathbb{N} \tag{5.2}$$

this is a consequence of the fact, demonstrated by Geroch in [17], that, in a family of spaces labeled by a continuous parameter, if the number of Killing vectors is independent on the label, in any limit for particular values of the label the number of Killing vectors must remain unchanged, but these can describe a different algebra. This is precisely our case: $\omega$ parametrizes a continuous set of spaces, all solutions to IIB equations, characterized by the same number of Killing vectors (namely $6+6+6+1 = 19$), then in the limit $\omega \to 0$ we must recover the same number of Killing vectors, but the algebra will no more be $\mathfrak{so}(2,2) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(2)$.

From a mathematical viewpoint those different rescalings correspond to an Inönü-Wigner contraction on the parent algebra and lead to the following set of generators:

$$\begin{aligned}
\xi_\pm &= -\partial_\pm & \xi_i &= -y_i \frac{\mu}{\sqrt{2}} \cos\frac{\mu u}{\sqrt{2}} \partial_- + \sin\frac{\mu u}{\sqrt{2}} \partial_i \\
\xi_{ij} &= -y_i\partial_j + y_j\partial_i & \xi_m &= -y_m \frac{\mu}{2} \cos\frac{\mu u}{2} \partial_- + \sin\frac{\mu u}{2} \partial_m
\end{aligned} \tag{5.3}$$

$$
\begin{aligned}
\xi_{mn} &= -y_m\partial_n + y_n\partial_m & \xi^*_i &= \cos\frac{\mu u}{\sqrt{2}}\partial_i + y_i\frac{\mu}{\sqrt{2}}\sin\frac{\mu u}{\sqrt{2}}\partial_- \\
\xi_q &= \partial_q & \xi^*_m &= \cos\frac{\mu u}{2}\partial_m + y_m\frac{\mu}{2}\sin\frac{\mu u}{2}\partial_- \\
\xi_{qr} &= -y_q\partial_r + y_r\partial_q \quad (q, r \neq 8)\,. & & \qquad (5.4)
\end{aligned}
$$

An important comment is in order: as we said in section 2, the SO(4) isometry of the metric (2.5) is apparent: the invariance under rotations holds only in the directions labeled by $(5\ldots7)$. Dealing now with the odd subalgebra, we can say that, since our solution preserves only one half of supersymmetry, $\mathfrak{G}_1$ has two generators, $Q^I$, $I = 1, 2$, possessing only one half of the degrees of freedom of usual SO(10) Majorana-Weyl spinors (they indeed satisfy the bound $\Gamma^+Q^I = 0$); the structure constants of their commutators with the even part of the supersymmetry algebra can be obtained using the spinorial Lie derivative, $\mathfrak{L}_\xi$ in the direction of the Killing vector $\xi$, which can be defined using the standard covariant derivative ($\nabla_M = \partial_M + \frac{1}{4}\omega_M^{\ PN}\Gamma_{PN}$) as:

$$
\mathfrak{L}_\xi\epsilon^I = \xi^M\nabla^{IJ}_M\epsilon + \frac{1}{4}\nabla^{IJ}_{[M}\xi_{N]}\Gamma^{MN}\epsilon^J\,. \qquad (5.5)
$$

It can be shown that, by construction, $\mathfrak{L}_\xi$ preserves the space of Killing spinors i.e. let $\xi$ be a Killing vector field; acting on a generic Killing spinor $\epsilon^I(\epsilon^I_+, \epsilon^I_-)$,[1] $\mathfrak{L}_\xi$ will give a Killing spinor with different parameters $\epsilon^I(S^+_\xi\epsilon^I_+, S^-_\xi\epsilon^I_-)$. This defines an action of the Lie algebra of isometries on the space of Killing spinors, whose structure constants are given by the matrices $S^+_\xi$ and $S^-_\xi$

$$
\mathfrak{L}_\xi\epsilon^I = S^{IJ}_\xi\epsilon^J\,. \qquad (5.6)
$$

Explicitly we can write:

$$
\begin{aligned}
\mathfrak{L}_+\epsilon^I &= \frac{1}{4}\mu\left(2W + \sqrt{2}Y\right)\sigma_3^{IJ}\epsilon^J & \mathfrak{L}_-\epsilon^I &= 0 \\
\mathfrak{L}_{ij}\epsilon^I &= \frac{1}{2}\Gamma_{ij}\epsilon^I & \mathfrak{L}_A\epsilon^I &= 0 \\
\mathfrak{L}_{mn}\epsilon^I &= \frac{1}{2}\Gamma_{mn}\epsilon^I & \mathfrak{L}^*_i\epsilon^I &= 0 \\
\mathfrak{L}_{qr}\epsilon^I &= \frac{1}{2}\Gamma_{qr}\epsilon^I\ (q, r \neq 8) & \mathfrak{L}^*_m\epsilon^I &= 0\,. \qquad (5.7)
\end{aligned}
$$

Finally, only the anticommutator of the odd generators remains to be computed, and this task can be easily achieved by looking at the nonzero terms in the fermionic product ($\chi^I$ and $\psi^J$ are two Killing spinors satisfying the same Killing equation):

$$
V = \bar{\chi}^I\Gamma^M\psi^J_2\partial_M \qquad (5.8)
$$

in fact, it is possible to demonstrate [18] that, if the two spinors satisfy Killing equation

[1] Any spinor can be split in a $+$ and a $-$ part as follows: $\epsilon = \eta^{+-}\epsilon = \frac{1}{2}\left\{\Gamma^+, \Gamma^-\right\}\epsilon = \frac{1}{2}\Gamma^+\Gamma^-\epsilon + \frac{1}{2}\Gamma^-\Gamma^+\epsilon$, so that $\Gamma^\pm\epsilon_\pm = 0$ since ${\Gamma^+}^2 = {\Gamma^-}^2 = 0$

with the same constant $\lambda$, $V$ must be a linear combination of Killing vectors, and the coefficients in front of them are the structure constants we are looking for; after some easy algebra, we can see that the only non-vanishing component of $V$ is:

$$V = -\bar{\chi}\Gamma^{-}\psi\xi_{-}\,. \tag{5.9}$$

Actually, $V$ can give us only the anticommutator of the various components of a single supersymmetry generator: there could be an antisymmetric part $\left\{Q^I_\alpha, Q^J_\beta\right\} = M_{\alpha\beta}\varepsilon^{IJ}$ with $M_{\alpha\beta}$ and $\varepsilon^{IJ}$ antisymmetric; in this case, $M$ should be a central charge, whose (anti)commutator with all the generators of the algebra vanish; but since in the parent algebra there was no such generator, and since no new generator can appear after a Penrose limit, we can conclude that this antisymmetric part does not appear in our algebra. Finally, we can write down the complete set of non-zero (anti)commutators of the superalgebra $\mathfrak{G} = \mathfrak{G}_0 \times \mathfrak{G}_1$:

$$\begin{array}{rclrcl}
[\xi_+, \xi_i] &=& -\dfrac{\mu}{\sqrt{2}}\xi^*_i & [\xi_{AB}, \xi_C] &=& -\eta_{AC}\xi_B + \eta_{BC}\xi_A \\
[\xi_+, \xi_m] &=& -\dfrac{\mu}{2}\xi^*_m & [\xi_{AB}, \xi^*_C] &=& -\eta_{AC}\xi^*_B + \eta_{BC}\xi^*_A \\
[\xi_+, \xi^*_i] &=& \dfrac{\mu}{\sqrt{2}}\xi_i & [\xi_{AB}, \xi_{CD}] &=& 4\eta_{[A_{[D}}\xi_{B]_{C]}} \\
[\xi_+, \xi^*_m] &=& \dfrac{\mu}{2}\xi_m & [\xi_+, Q^I] &=& \dfrac{1}{4}\mu\left(2W + \sqrt{2}Y\right)\sigma_3^{IJ}Q^J \\
[\xi^*_i, \xi_j] &=& \dfrac{\mu}{\sqrt{2}}\delta_{ij}\xi_- & [\xi_{AB}, Q^I] &=& \dfrac{1}{2}\Gamma_{AB}Q^I \\
[\xi^*_m, \xi_n] &=& \dfrac{\mu}{2}\delta_{mn}\xi_- & \left\{Q^I_\alpha, Q^J_\beta\right\} &=& -\left(\Gamma^- C^{-1}\right)_{\alpha\beta}\delta^{IJ}\xi_-\,,
\end{array} \tag{5.10}$$

where $C$ is the charge conjugation matrix and $\xi_{AB}$ is nonzero only when $A, B = \{1,2\}$ or $\{3,4\}$ or $\{5,6,7\}$.

## 6. Supersymmetric $\sigma$-model lagrangian

String theory is essentially a 2-dimensional field theory, whose fields $X^M$ and $\Theta^I$ are coupled via certain tensors (in IIB: a "metric" $g_{MN}$, and some "potentials" $A_{MN}$, $B_{\mathrm{MN}}$ and $F_{M_1\ldots M_5}$); in order for this theory to be well defined, we need Weyl invariance at the quantum level, and then we must impose a set of conditions on the "metric" and on the "potentials": it turns out that those conditions are exactly the supergravity field equations.

In other words, if we want a consistent string theory lagrangian, we must use background fields satisfying supergravity equations of motion. From now on, we will use upper case X to denote $\sigma$-model fields corresponding to target space coordinates. The $\sigma$-model lagrangian can be constructed using the metric $g_{\mathrm{MN}}$ and the potential 2-form $B$ defined, since the RR potential is set to zero, as:

$$\begin{aligned}
G &= dB \\
B_{uM} &= G_{uMN}X^N\,,
\end{aligned} \tag{6.1}$$



JHEP07(2003)035

where we used the gauge freedom to set $\partial_u B = 0$; in components:

$$\begin{aligned} B_{ui} &= 2\mu\varepsilon_{ij}X^j \\ B_{um} &= \sqrt{2}\mu\varepsilon_{mn}X^n \,. \end{aligned} \tag{6.2}$$

The supersymmetric $\sigma$-model lagrangian can be written according to the standard formula [4, 12]:

$$\begin{aligned} \mathcal{L} = \frac{1}{2}\left(\sqrt{-h}h^{ab}g_{MN} - \varepsilon^{ab}B_{MN}\right)\partial_a X^M \partial_b X^N + \\ + i\left(\sqrt{-h}h^{ab}\delta^{IJ} - \varepsilon^{ab}\sigma_3^{IJ}\right)\partial_a X^M \bar{\Theta}^I \Gamma_M \mathcal{D}_b^{JK}\Theta^K \,, \end{aligned} \tag{6.3}$$

where $h^{ab}$ is the world-sheet metric and $h$ its determinant, $\varepsilon$ the Levi-Civita tensor and $\sigma_{1,2,3}$ are the usual Pauli matrices; $\Theta^I$ (with $I = 1, 2$) are Majorana-Weyl 10 dimensional spinors, whose conjugate is, as usual, $\bar{\Theta}^I = \Theta^{I\dagger}\Gamma_0$, and the derivative operator $\mathcal{D}$ is defined as:

$$\begin{aligned} \mathcal{D}_a^{IJ} = \partial_a\delta^{IJ} + \frac{1}{4}\partial_a X^M \Bigg[\left(\omega_{M\ NP}\ \delta^{IJ} - \frac{1}{2}H_{MNP}\ \sigma_3^{IJ}\right)\Gamma^{NP} + \\ + \left(\frac{1}{3!}F^{RNP}\Gamma_{RNP}\sigma_1^{IJ} + \frac{i}{2\cdot 5!}F^{M_1\ldots M_5}\Gamma_{M_1\ldots M_5}\sigma_2^{IJ}\right)\Gamma_M\Bigg] \end{aligned} \tag{6.4}$$

that is, the pull-back of the generalized covariant derivative appearing in Killing spinor equation in IIB supergravity, where we divided the complex three-form $G$ into its NS and RR parts: $G = H + \Phi F$, where $\Phi$ is the dilaton-axion.

The lagrangian written above (6.3) is invariant under both rigid space-time supersymmetry transformations, depending on a constant parameter $\eta$:

$$\begin{aligned} \delta X^M &= -i\bar{\Theta}^I\Gamma^M\eta^I \\ \delta\Theta^I &= \eta^I \end{aligned} \tag{6.5}$$

and local supersymmetry (or $\kappa$-symmetry [19]) transformations which read [12]:

$$\begin{aligned} \delta\Theta^I &= (\mathbb{1} + \Gamma)^{IJ}\kappa^J \\ \delta X^M &= i\bar{\Theta}^I\Gamma^M\delta\Theta^I \,, \end{aligned} \tag{6.6}$$

where $\kappa$ is a spinor parametrizing this transformation, while the matrix $\Gamma$ is defined by:

$$\Gamma^{IJ} = \frac{1}{2\sqrt{-h}}\varepsilon^{ab}\partial_a X^M \partial_b X^N \Gamma_{MN}\sigma_3^{IJ} \,. \tag{6.7}$$

Turning back to the $\sigma$-model, we recall that the only nonzero components of the spin connection $\omega$ (3.4) and 3-form field $G$ (2.7) are:

$$\begin{aligned} \omega_{u\ +i} &= \frac{1}{2}\mu y_i = \frac{1}{2}\mu X_i \\ \omega_{u\ +m} &= \frac{1}{4}\mu y_m = \frac{1}{4}\mu X_m \end{aligned}$$

$$G_{uij} = 2\mu\varepsilon_{ij}$$
$$G_{umn} = \sqrt{2}\mu\varepsilon_{mn}\,. \tag{6.8}$$

Light-cone gauge consists of a set of conditions over both bosonic and fermionic fields, that is:

$$\begin{aligned} u &= \tau p^+\sqrt{\alpha'} \\ h^{ab} &= \eta^{ab} \\ \Gamma^+\Theta^I &= 0\,; \end{aligned} \tag{6.9}$$

where $p^+$ is a constant which may be interpreted as center of mass momentum of the string along the $u$ direction.

We can demonstrate that both pull-back terms give a nonzero contribution only along the $\tau$-$u$ direction, when $\partial_a X^M = \partial_\tau u = p^+\sqrt{\alpha'}$; first of all, let us examine the supercovariant derivative: the connection term is nonzero only when $M = u$ and the term $H_{MNP}\Gamma^{NP}$ always generates a $\Gamma^+$ when $M \neq u$; with these considerations, the general covariant derivative turns out to be:

$$\mathcal{D}_a^{IJ} = \partial_a\delta^{IJ} + \delta_a^0 Z\sigma_3^{IJ}\,, \tag{6.10}$$

where $Z$ can be expressed in terms of gamma matrices as:

$$Z = -\frac{1}{4}\mu p^+\sqrt{\alpha'}\left(2\Gamma^{12} + \sqrt{2}\Gamma^{34}\right)\,. \tag{6.11}$$

Let us consider instead the term $\bar{\Theta}^I\Gamma_M\mathcal{D}_b^{IJ}\Theta^J$: when $X^M = v$ it vanishes because of the light-cone gauge condition over $\Theta$, while when $M = A$ it vanishes because of the symmetry of Gamma matrices combined with the antisymmetry of grassmannian variables $\Theta^I$.

The final form of the lagrangian density in light-cone gauge is then:

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{2}\dot{X}^A\dot{X}_A - \frac{1}{2}X'^A X'_A - \frac{1}{4}\left(\mu p^+\right)^2\alpha'\left(X^i X_i + \frac{1}{2}X^m X_m\right) + \\ & + 2\mu p^+\sqrt{\alpha'}\varepsilon_{ij}X^i X'^j + \sqrt{2}\mu p^+\sqrt{\alpha'}\varepsilon_{mn}X^m X'^n + \\ & + ip^+\sqrt{\alpha'}\bar{\Theta}^1\Gamma^-\left(\partial_\tau - \partial_\sigma + Z\right)\Theta^1 + ip^+\sqrt{\alpha'}\bar{\Theta}^2\Gamma^-\left(\partial_\tau + \partial_\sigma - Z\right)\Theta^2\,, \end{aligned} \tag{6.12}$$

where dot and prime mean respectively derivative with respect to $\tau$ and $\sigma$. We can now rescale all the fields in $\mathcal{L}$ so that they all are adimensional (in practice, this amounts to sending $\mu \to \sqrt{\alpha'}\mu$ and considering also $\sigma$ and $\tau$ adimensional); moreover, we reabsorb the factor $p^+\sqrt{\alpha'}$ in the definition of fermions.

## 7. Spectrum of the hamiltonian

The procedure we adopt in order to find the spectrum of our theory, is the same in both fermionic and bosonic sectors: since we are dealing with a closed string theory, we expand all fields in Fourier modes and, by means of the equations of motion, write down the explicit dependence of fields on world sheet coordinates; then we promote Fourier coefficients to operators and impose canonical (anti)-commutation relations. The last step consists of writing explicitly $H$ in terms of creation and annihilation operators.



JHEP07(2003)035

### 7.1 Bosonic sector

Using Euler-Lagrange equations, we deduce the equations of motion for the $X$ fields, which are:

$$\begin{aligned} \ddot{X}^q &= X''^q & (7.1)\\ \ddot{X}^i &= -\frac{1}{2}\left(\mu p^+\alpha'\right)^2 X^i + X''^i - 4\left(\mu p^+\alpha'\right)\varepsilon^{ij}X'^j & (7.2)\\ \ddot{X}^m &= -\frac{1}{4}\left(\mu^2 p^+\alpha'\right)^2 X^m + X''^m - 2\sqrt{2}\left(\mu p^+\alpha'\right)\varepsilon^{mn}X'^n; & (7.3)\end{aligned}$$

since we are dealing with a closed string theory, we must take into account the periodicity condition on $\sigma$ coordinate ($X(\sigma+2\pi,\tau) = X(\sigma,\tau)$) and we can expand the $X$ fields in Fourier modes:

$$X^A(\sigma,\tau) = \sum_{s:-\infty}^{+\infty} \mathcal{C}_s^A(\tau)e^{is\sigma} \tag{7.4}$$

because of the reality condition that must be imposed on all the $X$ fields, Fourier coefficients in (7.4) must obey:

$$\mathcal{C}_s^A(\tau) = \mathcal{C}_{-s}^{A\;*}(\tau) \tag{7.5}$$

now equations (7.1), (7.2) and (7.3) can be rewritten as a single matricial relation among coefficients, namely:

$$\ddot{\mathcal{C}}_s^A = -T_s{}^A{}_B\,\mathcal{C}_s^B \quad \text{(no sum over } s\text{)}\,, \tag{7.6}$$

where matrix $T_s$ is defined to be ($\lambda = \mu p^+\alpha'$):

$$T_s = \begin{pmatrix} \frac{1}{2}\lambda^2+s^2 & -4i\lambda s & 0 & 0 & 0 & 0 & 0 & 0\\ 4i\lambda s & \frac{1}{2}\lambda^2+s^2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{4}\lambda^2+s^2 & -2\sqrt{2}i\lambda s & 0 & 0 & 0 & 0\\ 0 & 0 & 2\sqrt{2}i\lambda s & \frac{1}{4}\lambda^2+s^2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & s^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & s^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & s^2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s^2 \end{pmatrix} \tag{7.7}$$

whose eigenvalues are:

$$\begin{aligned} & s^2 \quad \text{(4 times)}\\ k_s^\pm &= \frac{1}{2}\left(2s^2 \pm 8s\mu p^+\alpha' + \left(\mu p^+\alpha'\right)^2\right)\\ j_s^\pm &= \frac{1}{4}\left(4s^2 \pm 8\sqrt{2}s\mu p^+\alpha' + \left(\mu p^+\alpha'\right)^2\right) \end{aligned} \tag{7.8}$$

in order to find a solution to equation (7.6) it is convenient to introduce a set of orthonormal complex eigenvectors of $T_s$ ($v^\pm$ and $w^\pm$, obeying $v^\mp\cdot v^\pm = w^\mp\cdot w^\pm = 1$, all other products being 0), corresponding respectively to eigenvalues $k^\pm$ and $j^\pm$ in (7.8), and reorganize the $X^i$ and $X^m$ fields as vectors along $v^\pm$ and $w^\pm$ basis:

$$X^{(i)} = \begin{pmatrix} X^1\\ X^2 \end{pmatrix} = \sum_{s=-\infty}^{+\infty} \mathcal{C}_s^{(i)} e^{is\sigma}$$

$$X^{(m)} = \begin{pmatrix} X^3 \\ X^4 \end{pmatrix} = \sum_{s=-\infty}^{+\infty} \mathcal{C}_s^{(m)} e^{is\sigma} \,, \tag{7.9}$$

where, for every $s$:

$$\begin{aligned} \mathcal{C}_s^{(i)} &= \begin{pmatrix} \mathcal{C}^1 \\ \mathcal{C}^2 \end{pmatrix} = c_s^+ v^+ + c_s^- v^- \\ \mathcal{C}_s^{(m)} &= \begin{pmatrix} \mathcal{C}^3 \\ \mathcal{C}^4 \end{pmatrix} = d_s^+ w^+ + d_s^- w^- \end{aligned} \tag{7.10}$$

now, the reality condition on the $c_s^\pm$ and $d_s^\pm$ read:

$$\begin{array}{lllll} c^{\mp}_{-s}{}^* v^{\mp *} = c^{\mp}_{-s}{}^* v^{\pm} & = c_s^\pm v^\pm & \Rightarrow & c^{\mp}_{-s}{}^* = c_s^\pm \\ d^{\mp}_{-s}{}^* w^{\mp *} = d^{\mp}_{-s}{}^* w^{\pm} & = d_s^\pm w^\pm & \Rightarrow & d^{\mp}_{-s}{}^* = d_s^\pm \end{array} \tag{7.11}$$

and the equations of motion, which are greatly simplified with these conventions, now read:

$$\begin{aligned} \ddot{\mathcal{C}}_s^q &= -s^2 \mathcal{C}_s^q \\ \ddot{c}_s^\pm &= -k_s^\pm c_s^\pm \\ \ddot{d}_s^\pm &= -j_s^\pm d_s^\pm \end{aligned} \tag{7.12}$$

and can be easily solved in terms of the square roots of $T$ eigenvalues $|s|$, $\omega_s^\pm = \sqrt{k_s^\pm}$ and $\nu_s^\pm = \sqrt{j_s^\pm}$:

$$\begin{aligned} \mathcal{C}_0^q &= x^q + p^q \tau \\ \mathcal{C}_s^q &= A_s^q e^{-i|s|\tau} + B_s^q e^{i|s|\tau} \\ c_s^\pm &= A_s^\pm \; e^{-i\omega_s^\pm \tau} + B_s^\pm \; e^{i\omega_s^\pm \tau} \\ d_s^\pm &= C_s^\pm \; e^{-i\nu_s^\pm \tau} + D_s^\pm \; e^{i\nu_s^\pm \tau} \,. \end{aligned} \tag{7.13}$$

Reality condition can be written in terms of the new coefficients, and become:

$$\begin{aligned} B^q_{-s}{}^* &= A_s^q \\ B^{\mp}_{-s}{}^* &= A_s^\pm \\ D^{\mp}_{-s}{}^* &= C_s^\pm \end{aligned} \tag{7.14}$$

we can write down $X$ fields as:

$$\begin{aligned} X^q &= x^q + p^q \tau + \sum_{s \neq 0} \left( A_s^q e^{-i|s|\tau} + A^q_{-s}{}^* e^{i|s|\tau} \right) e^{is\sigma} \\ X^{(i)} &= \sum_{s=-\infty}^{+\infty} \left[ \left( A_s^+ e^{-i\omega_s^+ \tau} + A^-_{-s}{}^* e^{i\omega_s^+ \tau} \right) v^+ + \left( A_s^- e^{-i\omega_s^- \tau} + A^+_{-s}{}^* e^{i\omega_s^- \tau} \right) v^- \right] e^{is\sigma} \\ X^{(m)} &= \sum_{s=-\infty}^{+\infty} \left[ \left( C_s^+ e^{-i\nu_s^+ \tau} + C^-_{-s}{}^* e^{i\nu_s^+ \tau} \right) w^+ + \left( C_s^- e^{-i\nu_s^- \tau} + C^+_{-s}{}^* e^{i\nu_s^- \tau} \right) w^- \right] e^{is\sigma} \end{aligned} \tag{7.15}$$

equipped with these definitions, we can write the momenta conjugate to target space coordinates $X$:

$$\Pi_A = \frac{\delta L}{\delta \dot{X}^A} = \dot{X}_A \tag{7.16}$$

which explicitly read:

$$\begin{aligned}
\Pi^q &= p^q - \sum_{s\neq 0} i|s| \left( A^q_s e^{-i|s|\tau} - {A^q_{-s}}^* e^{i|s|\tau} \right) e^{is\sigma} \\
\Pi^{(i)} &= \sum_{s=-\infty}^{+\infty} -i \left[ \omega^\pm_s \left( A^\pm_s e^{-i\omega^\pm_s \tau} - {A^\mp_{-s}}^* e^{i\omega^\pm_s \tau} \right) v^\pm \right] e^{is\sigma} \\
\Pi^{(m)} &= \sum_{s=-\infty}^{+\infty} -i \left[ \nu^\pm_s \left( C^\pm_s e^{-i\nu^\pm_s \tau} - {C^\mp_{-s}}^* e^{i\nu^\pm_s \tau} \right) w^\pm \right] e^{is\sigma}
\end{aligned} \tag{7.17}$$

it is possible to invert the above relations and get an expression for the Fourier coefficients:

$$\begin{aligned}
x^q &= \int_0^{2\pi} \frac{d\sigma}{2\pi} \left( X^q - \tau \Pi^q \right) \\
p^q &= \int_0^{2\pi} \frac{d\sigma}{2\pi} \Pi^q \\
A^q_s &= \frac{1}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left( X^q + \frac{i}{|s|} \Pi^q \right) e^{i(|s|\tau - s\sigma)} \\
A^\pm_s &= \frac{1}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left( X^{(i)} + \frac{i}{\omega^\pm_s} \Pi^{(i)} \right) \cdot v^\mp e^{i(\omega^\pm_s \tau - s\sigma)} \\
C^\pm_s &= \frac{1}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left( X^{(m)} + \frac{i}{\nu^\pm_s} \Pi^{(m)} \right) \cdot w^\mp e^{i(\nu^\pm_s \tau - s\sigma)}
\end{aligned} \tag{7.18}$$

in order to quantize this lagrangian, we promote $X$ and $\Pi$ to operators, imposing canonical commutation relations:

$$\left[ X^A(\tau, \sigma), \Pi_B(\tau, \sigma') \right] = i \delta^A_B \delta(\sigma - \sigma') \,. \tag{7.19}$$

Condition (7.19) imply that, when we interpret Fourier coefficients as operators ($A^* \to A^\dagger$):

$$\begin{aligned}
\left[ x^q, p^{q'} \right] &= \frac{i}{4\pi} \eta^{qq'} \\
\left[ A^q_s, {A^{q'}_{s'}}^\dagger \right] &= \frac{1}{8\pi|s|} \delta_{ss'} \eta^{qq'} \\
\left[ A^\pm_s, {A^\pm_{s'}}^\dagger \right] &= \frac{1}{8\pi \omega^\pm_s} \delta_{ss'} \\
\left[ C^\pm_s, {C^\pm_{s'}}^\dagger \right] &= \frac{1}{8\pi \nu^\pm_s} \delta_{ss'}
\end{aligned} \tag{7.20}$$

all the above operators can be rescaled in order to have canonical commutation relations for creation and annihilation operators:

$$\hat{x}^q \ , \ \hat{p}^q = 2\sqrt{\pi}\, x^q \ , \ p^q$$

$$
\begin{aligned}
a_s^q &= 2\sqrt{2\pi|s|}A_s^q \\
a_s^{\pm} &= 2\sqrt{2\pi\omega_s^{\pm}}A_s^{\pm} \\
c_s^{\pm} &= 2\sqrt{2\pi\nu_s^{\pm}}C_s^{\pm}\,.
\end{aligned}
\tag{7.21}
$$

Now we can construct the hamiltonian, with a Legendre transformation:

$$
H_{\mathrm{bos}} = \int d\sigma\left(\Pi_A X^A - \mathcal{L}_{\mathrm{bos}}\right)
\tag{7.22}
$$

after some algebra, we can write an explicit expression for $H$ in terms of $X$ and $\Pi$:

$$
\begin{aligned}
H_{\mathrm{bos}} &= H_{\mathrm{bos}}^{(q)} + H_{\mathrm{bos}}^{(i)} + H_{\mathrm{bos}}^{(m)} \\
H_{\mathrm{bos}}^{(q)} &= \frac{1}{2}\dot{X}^q\dot{X}_q + \frac{1}{2}X'^q X'_q \\
H_{\mathrm{bos}}^{(i)} &= \frac{1}{2}\dot{X}^i\dot{X}_i + \frac{1}{2}X'^i X'_i + \frac{1}{4}\left(\mu p^+\alpha'\right)^2 X^i X_i - 2\mu p^+\alpha'\varepsilon_{ij}X^i X'^j \\
H_{\mathrm{bos}}^{(m)} &= \frac{1}{2}\dot{X}^m\dot{X}_m + \frac{1}{2}X'^m X'_m + \frac{1}{8}\left(\mu p^+\alpha'\right)^2 X^m X_m - 2\sqrt{2}\mu p^+\alpha'\varepsilon_{mn}X^m X'^n
\end{aligned}
\tag{7.23}
$$

using now the definition of $X$ in terms of creation and annihilation operators, we can write $H$ in terms of number operators, namely:

$$
\begin{aligned}
H_{\mathrm{bos}}^{(q)} &= \frac{1}{4}\hat{p}^q\hat{p}_q + \sum_{s=-\infty}^{+\infty}\frac{|s|}{4}\sum_q\left(2\mathcal{N}_s^q + 1\right) \\
H_{\mathrm{bos}}^{(i)} &= \sum_{s=-\infty}^{+\infty}\left[\omega_s^{\pm}\left(2\mathcal{N}_s^{(i)\ \pm} + 1\right)\right] \\
H_{\mathrm{bos}}^{(m)} &= \sum_{s=-\infty}^{+\infty}\left[\nu_s^{\pm}\left(2\mathcal{N}_s^{(m)\ \pm} + 1\right)\right],
\end{aligned}
\tag{7.24}
$$

where

$$
\begin{aligned}
\mathcal{N}_s^q &= a_s^{q\ \dagger}a_{s\ q} \\
\mathcal{N}_s^{(i)\ \pm} &= -a_s^{(i)\ \pm\ \dagger}\cdot a_s^{(i)\ \pm} \\
\mathcal{N}_s^{(m)\ \pm} &= -c_s^{(m)\ \pm\ \dagger}\cdot c_s^{(m)\ \pm}
\end{aligned}
\tag{7.25}
$$

as it was expected, the hamiltonian is proportional to number operators. There is one more condition on these states, which can be obtained imposing that a translation in $\sigma$ does not affect the physical result; the operator generating $\sigma$-translations is:

$$
P = \int d\sigma \Pi^A \partial_\sigma X_A\,.
\tag{7.26}
$$

After some algebra on creation and annihilation operators, we obtain that physical states must obey:

$$
\mathcal{N}^{q\ R} = \mathcal{N}^{q\ L}
$$

$$\mathcal{N}^{(i)\ +} = \mathcal{N}^{(i)\ -}$$
$$\mathcal{N}^{(m)\ +} = \mathcal{N}^{(m)\ -}\,, \tag{7.27}$$

where $\mathcal{N}^{q\ R,L}$ represents the total number of left- ($n > 0$) or right-movers ($n < 0$) in the flat sector, while $\mathcal{N}^{(i),(m)\ \pm} = \sum_s \mathcal{N}_s^{(i),(m)\ \pm}$ represents the total number of $(i)$ or $(m)$ "left" $(+)$ or "right" $(-)$ modes: it is again a sort of *level-matching* condition.

### 7.2 Fermionic sector

We start with the fermionic equations of motion, which read:

$$(\partial_\tau - \partial_\sigma + Z)\,\Theta^1 = 0$$
$$(\partial_\tau + \partial_\sigma - Z)\,\Theta^2 = 0\,. \tag{7.28}$$

In order to solve these equations, we expand the fermionic fields $\Theta^I$ in Fourier modes, taking into account periodicity condition over $\sigma$ and reality of both spinors:

$$\Theta^I = \sum_{n:-\infty}^{+\infty} \Theta^I_n(\tau) e^{in\sigma}$$
$$\Theta^{I\ *}_{-n} = \Theta^I_n \tag{7.29}$$

it is important to notice that, because of the two conditions (light-cone and Weyl) we imposed on $\Theta$, these spinor fields have only 8 nonzero components (and we may choose them to be the upper ones), such that (see appendix A):

$$\Theta^I = \begin{pmatrix} S^I \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \Theta^I_n = \begin{pmatrix} S^I_n \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7.30}$$

Since the matrix $Z$ has a block-diagonal form, we can simplify equations of motion as (see appendix A for conventions on gamma matrices):

$$\left[\partial_\tau \delta^{IJ} - \left(\partial_\sigma + \frac{1}{4}\left(\mu p^+ \alpha'\right)\left(\sqrt{2}c^{34} + 2c^{12}\right)\right)\sigma_3^{IJ}\right] S^I = 0 \tag{7.31}$$

like in the bosonic case, we introduce a set of orthonormal eigenvectors $u_\alpha$ ($\alpha$ runs over spinorial indices $1\ldots 8$, and $u^*_\alpha \cdot u_\beta = \delta_{\alpha\beta}$) of the matrix $M_n = n\mathbb{1} + iZ$ , such that:

$$S^I_n = \sum_{\alpha=1}^{8} \zeta^I_{n\ \alpha}(\tau)\ u_\alpha$$
$$M_n u_\alpha = \omega_{n\ \alpha} u_\alpha \text{ (no sum on } \alpha)\,, \tag{7.32}$$

where the $\omega_{n\ \alpha}$ are:

$$\omega_{n\ 1} = \omega_{n\ 2} = \left[n - \frac{1}{4}\mu p^+ \alpha'(2 + \sqrt{2})\right]$$

$$
\begin{aligned}
\omega_{n\ 3} = \omega_{n\ 4} &= \left[ n + \frac{1}{4}\mu p^+ \alpha'(2-\sqrt{2}) \right] \\
\omega_{n\ 5} = \omega_{n\ 6} &= \left[ n + \frac{1}{4}\mu p^+ \alpha'(\sqrt{2}-2) \right] \\
\omega_{n\ 7} = \omega_{n\ 8} &= \left[ n + \frac{1}{4}\mu p^+ \alpha'(2+\sqrt{2}) \right]
\end{aligned}
\tag{7.33}
$$

and the following relations among eigenvectors and eigenvalues hold:

$$
\begin{aligned}
{u_\alpha}^* &= u_{(9-\alpha)} \\
\omega_{-n\ (9-\alpha)} &= -\omega_{n\ \alpha}
\end{aligned}
\tag{7.34}
$$

so that the reality condition becomes:

$$
\left( \zeta^I_{-n\ (9-\alpha)} \right)^* = \zeta^I_{n\ \alpha} \tag{7.35}
$$

now the equations of motion read:

$$
\dot{\zeta}^I_{n\ \alpha} = i\sigma_3^{IJ} \omega^\alpha_n \zeta^J_{n\ \alpha} \ (\text{no sum over } \alpha) \tag{7.36}
$$

which can be easily integrated:

$$
\zeta^I_{n\ \alpha} = \psi^I_{n\ \alpha} e^{i\sigma_3^{II}\omega^\alpha_n \tau} . \tag{7.37}
$$

Finally, spinors $S^I$ can be written as:

$$
S^I = \sum_{n=-\infty}^{+\infty} \sum_{\alpha=1}^{8} \psi^I_{n\ \alpha} e^{i\sigma_3^{II}\omega^\alpha_n \tau} e^{in\sigma} u_\alpha \tag{7.38}
$$

this relation can be inverted to obtain Fourier coefficients:

$$
\psi^I_{m\ \beta} = \int \frac{d\sigma}{2\pi} e^{-im\sigma} u^*_\beta \cdot S^I e^{-i\sigma_3^{II}\omega^\beta_m \tau} \tag{7.39}
$$

as usual in IIB string theory $\sigma$-model, the two spinors correspond to right- and left-movers along the closed string, depending on the sign of $m$; we can now find the momenta conjugate to fermionic coordinates, which are:

$$
\pi^I = \frac{\delta \mathcal{L}}{\delta \partial_\tau S^I} = -i {S^I}^* \Gamma^- \qquad \bar{\pi}^I = \frac{\delta \mathcal{L}}{\delta \partial_\tau \bar{S}^I} = 0 \tag{7.40}
$$

using these definitions, we can construct the fermionic part of the hamiltonian $H_{\text{fermi}}$ which turns out to be:

$$
\begin{aligned}
H_{\text{fermi}} &= \int d\sigma \left( \pi^I \partial_\tau S^I + \partial_\tau {S^I}^* \bar{\pi}^I - \mathcal{L}_{\text{fermi}} \right) \\
&= i \int d\sigma \left[ {S^I}^* \Gamma^- \left( \partial_\sigma - Z \right) \sigma_3^{IJ} S^J \right]
\end{aligned}
\tag{7.41}
$$

with the help of the equations of motion (7.28), we can simplify the above expression, getting:

$$H_{\rm fermi} = i \int d\sigma\ S^{I^*}\Gamma^- \partial_\tau S^I\ . \tag{7.42}$$

In order to quantize this hamiltonian, we impose canonical anticommutation relations on fermionic fields, which read:

$$\begin{aligned} \left\{ S^I_\alpha(\sigma,\tau), S^{J\dagger}_\beta(\sigma',\tau)\right\} &= \delta^{IJ}\delta_{\alpha\beta}\delta(\sigma-\sigma') \\ S^I_\alpha &= u^*_\alpha \cdot S^I \end{aligned} \tag{7.43}$$

these conditions imply analogue ones over operators $\psi$ and $\psi^*$:

$$\left\{ \psi^I_{n\ \alpha}, {\psi^J_{m\ \beta}}^* \right\} = \frac{1}{2\pi}\delta_{mn}\delta_{\alpha\beta}\delta^{IJ} \tag{7.44}$$

we can define creation and annihilation operators, by means of the following relations:

$$\begin{aligned} \xi^I_{m\ \alpha} &= \sqrt{2\pi}{\psi^I_{m\ \alpha}}^* \\ {\xi^I_{m\ \alpha}}^\dagger &= \sqrt{2\pi}\psi^I_{m\ \alpha} \end{aligned} \tag{7.45}$$

correspondingly, the hamiltonian becomes:

$$H_{\rm fermi} = \sum_\alpha \left[ \sum_I \sum_{n=-\infty}^{+\infty} \omega_{n\ \alpha}\left(2\mathcal{N}^I_{n\ \alpha} - 1\right)\right], \tag{7.46}$$

where as usual:

$$\mathcal{N}^I_{n\ \alpha} = {\xi^I_{n\ \alpha}}^\dagger \xi^I_{n\ \alpha}\,. \tag{7.47}$$

It is worthwhile to stress again the fact that $\sigma$-model coming from pp-wave type solutions to IIB supergravity is solvable, being quadratic in both fermionic and bosonic fields in light-cone gauge. Another interesting remark is in order: despite the solution has supersymmetries, the masses of bosonic and fermionic particles are different, and hence the string has a non null zero-point energy (analogous features in different backgrounds were seen in [20, 21]), as we can see from the structure of the eigenvalues of the hamiltonian:

$$\begin{aligned} \omega^{\rm bos}_n &= \sqrt{n^2 + an\left(\mu p^+\alpha'\right) + b\left(\mu p^+\alpha'\right)^2} \\ \omega^{\rm fermi}_n &= n + \frac{1}{4}c\left(\mu p^+\alpha'\right), \end{aligned} \tag{7.48}$$

where the coefficients $a$, $b$ and $c$ take the values for the fields $X^A$ and the components of $\Theta^I$:

$$\begin{aligned} a &= \left(0,0,0,0,4,4,2\sqrt{2},2\sqrt{2}\right) \\ b &= \left(0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4}\right) \\ c &= \left(-2-\sqrt{2},-2-\sqrt{2},2-\sqrt{2},2-\sqrt{2},-2+\sqrt{2},-2+\sqrt{2},2+\sqrt{2},2+\sqrt{2}\right). \end{aligned}$$

The reason for this asymmetry is easy to explain: when we impose fermionic gauge conditions, ($\Gamma^+\Theta = 0$) we are fixing a certain amount of supersymmetry; the residual supersymmetry transformations should then preserve not only the equations of motion (and this is guaranteed by the explicit supersymmetric construction of the lagrangian), but also the gauge fixing conditions. It can be demonstrated (see [12]) that half of the components of the parameter $\kappa^I$ should be used as compensator of $\epsilon^I$ variation in order to maintain condition $\Gamma^+\Theta = 0$; then it turns out that light-cone condition $X^+ = p^+\sqrt{\alpha'}\tau$ is always satisfied; the residual supersymmetry transformations preserving both the lagrangian and the gauge conditions can be divided into two sets: there are shift (inhomogeneous) supersymmetries, involving only the fermionic fields and depending on Killing vectors $\epsilon^I$ such that $\Gamma^+\epsilon^I = 0$, and homogeneous supersymmetries, that have the usual form and depend on parameters $\epsilon^I$ such that $\Gamma^-\epsilon^I = 0$: the so-called "supernumerary" Killing spinors. Notice that this asymmetry in the masses is only due to the gauge fixing: we could have chosen, for instance, the so-called physical gauge ($X^0 = \tau$, $X^9 = \sigma$ and $\Gamma_{1\ldots 8}\Theta = 0$) [12], where the hamiltonian is no more easily quantizable, but the supersymmetries are explicit in the model.

## 8. BMN duality

BMN duality consists of a particular limit on both sides of $AdS/CFT$ correspondence: on the gravity side, it corresponds to a Penrose limit, which permits us to get a quantizable background of IIB string theory; on the CFT side, it corresponds to a large $J$ and $\Delta$ limit, with finite $\Delta - J$: the precise statement by BMN is that there exists a certain set of non-BPS operators whose two-point function reproduces exactly the string spectrum. In order to test this conjecture, the first step consists in identifying the CFT living on the border of $AdS$; the $AdS_3 \times S_3 \times S_3 \times S_1$ configuration can be obtained as a near horizon limit from a double system of $D1 - D5$-branes characterized by:

$$N_5^{(1)} = N_5^{(2)} = N_5 \,, \tag{8.1}$$

where $N_a^{(i)}$ is the number of $Da$-branes of the $i$ system; the dilaton is set to a constant

$$e^{-\phi} = \sqrt{2} \tag{8.2}$$

and $N_5$ is related to the number of parallel coincident $D1$-branes:

$$N_5 = N_1 = \sqrt{2N_1^{(1)}N_1(^2)} \tag{8.3}$$

moreover, the radii of the 3-spheres are proportional to the number of $D1$-branes:

$$\begin{aligned} R_{AdS} &= \sqrt{\frac{1}{2}\alpha' N_1} \\ R_{\rm Sph} &= \sqrt{\alpha' N_1} \end{aligned} \tag{8.4}$$

while the radius of the $S^1$ is not fixed by the configuration; as shown in [22], the symmetry algebra of the $AdS$ background is

$$\mathrm{SO}(2,2) \times \mathrm{SO}(4) \times \mathrm{SO}(4) \times \mathrm{U}(1) \tag{8.5}$$

which can be rewritten as:

$$(\mathrm{Sl}(2,R) \times \mathrm{SU}(2) \times \mathrm{SU}(2))^2 \times \mathrm{U}(1)\,. \tag{8.6}$$

The dual CFT can be shown [23] to be one of the $\mathcal{N} = (4,4)$ double SCFT based on the $A_\gamma$ algebras, found in [24], which are the Kac-Moody extension of (8.6). $A_\gamma$ algebras are characterized by two independent parameters: either the levels of the two affine SU(2)'s or the central charge and $\gamma$, which measures the relative weight of the levels $k^\pm$; since the levels $k^\pm$ are related to the radii of the two 3-spheres, we can conclude that $k^+ = k^-$. Actually, the levels of the affine algebras $\hat{\mathrm{SU}}(2)$ are:

$$k^\pm = \frac{R^2_{\mathrm{Sph}}}{4R_{AdS}G^{(3)}_{\mathrm{N}}} = \frac{4R^7 L}{\pi g^2 \alpha'^4} \tag{8.7}$$

and are related to the central charge $c$ and to $\gamma$ by the relations:

$$\begin{aligned} c &= \frac{6k^+k^-}{k^+ + k^-} \\ k &= k^+ + k^- = \frac{c}{6\gamma\,(1-\gamma)}. \end{aligned} \tag{8.8}$$

The central charge is instead given by the formula:

$$c = \frac{3R_{AdS}}{2G^{(3)}_{\mathrm{N}}}\,, \tag{8.9}$$

where $G^{(3)}_{\mathrm{N}}$ is the three-dimensional Newton constant, which can be obtained as:

$$G^{(3)}_{\mathrm{N}} = \frac{G^{(10)}_{\mathrm{N}}}{\mathrm{Vol}(S_3 \times S_3 \times S_1)} \tag{8.10}$$

with

$$\mathrm{Vol}(S^3) = 2\pi^2 R^3_{\mathrm{Sph}}\,, \qquad \mathrm{Vol}(S^1) = 2\pi L \tag{8.11}$$

and $L$ is the radius of $S^1$; we can conclude that

$$G^{(10)}_{\mathrm{N}} = 8\pi^6 g^2 \alpha'^4 \quad \rightarrow \quad G^{(3)}_{\mathrm{N}} = \frac{\pi g^2 \alpha'^4}{8R^6 L} \tag{8.12}$$

and then:

$$c = 12\frac{R^7 L}{\pi g^2 \alpha'^4} \tag{8.13}$$

from which we can conclude that $\gamma = 1/2$.

The $A_\gamma$ algebras are defined via the OPE's of their generators [25]:

$$\begin{aligned} T(z)T(w) &= \frac{c}{2(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots \\ T(z)\phi(w) &= \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + \cdots \end{aligned}$$

$$
\begin{aligned}
G_a(z)G_b(w) &= \frac{2c\delta_{ab}}{3(z-w)^3} + \frac{2M_{ab}(w)}{(z-w)^2} + \frac{(2T(w)\delta_{ab} + \partial M_{ab}(w))}{(z-w)} + \cdots \\
A^{\pm i}(z)G_a(w) &= \alpha^{\pm i}{}_a{}^b \frac{G_b(w)}{(z-w)} \mp \frac{2k^{\pm}Q_b(w)}{k(z-w)^2} + \cdots \\
A^{\pm i}(z)A^{\pm j}(w) &= \frac{\epsilon^{ijk}A^{\pm}_k(w)}{(z-w)} - \frac{k^{\pm}\delta^{ij}}{2(z-w)^2} + \cdots \\
Q_a(z)G_b(w) &= \frac{2\left(\alpha^{+i}_{ab}A^+_i(w) - \alpha^{-i}_{ab}A^-_i(w)\right)}{(z-w)} + \frac{\delta_{ab}U(w)}{(z-w)} + \cdots \\
A^{\pm i}(z)Q_a(w) &= \alpha^{\pm i}{}_a{}^b \frac{Q_b(w)}{(z-w)} + \cdots \\
U(z)G_a(w) &= \frac{Q_a(w)}{(z-w)^2} + \cdots \\
Q_a(z)Q_b(w) &= -\frac{k\delta_{ab}}{2(z-w)} + \cdots \\
U(z)U(w) &= -\frac{k}{2(z-w)^2} + \cdots ,
\end{aligned}
\tag{8.14}
$$

where $\phi$ stands for all the fields $G$, $A$, $U$ and $Q$ which have conformal weight $h$ equal to 3/2, 1, 1 and 1/2 respectively; in complex notations $i = \{+,-,3\}$, $a = \{+,-,+K,-K\}$ and the non-vanishing values (up to symmetry) of the various symbols are

$$
\begin{aligned}
&\delta_{+-} = \delta_{+K\,-K} = \frac{1}{2} && \epsilon^{+-}{}_3 = -2i && \epsilon^{3\pm}{}_{\pm} = \mp i \\
&\alpha^{\pm 3}_{+-} = -\frac{i}{4} && \alpha^{\pm 3}_{+K\,-K} = \mp\frac{i}{4} && \alpha^{+-}_{+\,+K} = \frac{i}{2} \\
&\alpha^{++}_{-\,-K} = -\frac{i}{2} && \alpha^{-+}_{-\,+K} = -\frac{i}{2} && \alpha^{--}_{+\,-K} = \frac{i}{2} \\
&M_{ab} \equiv -\frac{4}{k}\left[k^-\alpha^{+i}_{ab}A^+_i + k^+\alpha^{-i}_{ab}A^-_i\right]
\end{aligned}
\tag{8.15}
$$

The explicit study of BMN conjecture in this framework is postponed to a future work, but a brief algebraic analysis can help us in guessing some feature of BMN duality in this case: we need to identify states which have large R-charge $J$ and conformal dimension $\Delta$. In the case at hand, $J$ can be seen as a U(1) generator in the $\mathrm{SU}(2)^+_1 \oplus \mathrm{SU}(2)^+_2$ algebra related to first of the 3-spheres: from this point of view, the bosonic fields that are "rotated" by the first SO(4) have a well defined non-zero R-charge $J$, and correspond respectively to $Z$ and $\phi$ fields in [5] while the other four fields have by definition $J = 1$, since they belong to the second SO(4) and hence are not rotated so that they are not in relation with BMN operators in CFT, that is, they do not receive corrections to conformal dimension; this is in perfect agreement with string theory spectrum, since in 4 directions we have flat space spectrum $(\Delta - J)_s = s$, while in the others we have the expected deviations.

## 9. Conclusions

In summary, we found the Penrose limit of a particular background of IIB supergravity (2.1) and (2.2), and studied its behaviour, which has proven to be the expected one, being a Cahen-Wallach space [15]. Then we showed that in this background there is no supersymmetry enhancement, since only one half of the supersymmetry charges are conserved as in the parent background [11]. We also found the generators of the superalgebra of the pp-wave solution and their commutators and showed that the algebra can be written as $\mathfrak{h}(4) \rtimes (\mathfrak{so}(2) \oplus \mathfrak{so}(2)) \oplus \mathfrak{cso}(4) \oplus \mathfrak{so}(2)$, where $\mathfrak{h}$ is a Heisenberg algebra while $\mathfrak{cso}$ stands for a contracted $\mathfrak{so}$ algebra. We built the supersymmetric $\sigma$-model lagrangian and quantized the corresponding hamiltonian showing that it is solvable, being quadratic in both fermionic and bosonic fields; moreover, we found that in this model, being the number of supersymmetry charges the minimum for such a solution, i.e. there are no "supernumerary" Killing spinors, the masses of fermionic fields differ from those of the bosonic ones [12], and this leads to a non null zero-point energy. In the last chapter, a brief comment on BMN duality is given, discussing the CFT algebra in relation to string theory spectrum; a complete and explicit analysis remains still to be performed, but on algebraic grounds we showed that some expected behaviour holds.

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## A. Conventions and gamma matrices

We summarize the conventions adopted in this paper:

- $a$,$b$ (world sheet indices corresponding to $\sigma$ and $\tau$) run over 0,1
- $M$,$N$ run from 0 to 9
- $A$,$B$ from 1 to 8
- $i$,$j$ from 1 to 2 (labeling the coordinates trivially involved in the limit in $AdS$)
- $m$,$n$ from 3 to 4 (labeling the coordinates trivially involved in the limit in the first $S^3$)
- $q$,$r$ from 5 to 8 (labeling the coordinates in the last $S^3$ and $S^1$ after the limit)
- $I$,$J$ from 1 to 2 (labeling fermionic fields $\Theta$)
- $\alpha$, $\beta$ from 1 to 8 (spinorial indices).

The world-sheet metric $\eta_{ab}$ is:

$$\eta_{00} = -\eta_{11} = 1 \tag{A.1}$$

while tangent space metric is taken to be "mostly minus":

$$\begin{aligned}\eta_{++} &= \eta_{--}=0\\ \eta_{+-} &= \eta_{-+}=1\\ \eta_{AB} &= -\delta_{AB}\end{aligned} \tag{A.2}$$

the Levi-Civita tensor is:

$$\varepsilon^{01}=-\varepsilon^{10}=1 \tag{A.3}$$

and the $\sigma$ are the usual Pauli matrices; for the gamma matrices, we adopt a convention similar to that in [3], that is:

$$\Gamma_0=\begin{pmatrix}0 & \gamma_9\\ \gamma_9 & 0\end{pmatrix}, \qquad \Gamma_A=\begin{pmatrix}0 & \gamma_A\\ \gamma_A & 0\end{pmatrix}, \qquad \Gamma_9=\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}, \tag{A.4}$$

where $\gamma_A$ are SO(8) gamma matrices obeying:

$$\{\gamma_A,\gamma_B\}=-2\delta_{AB} \tag{A.5}$$

while $\gamma_9$ is defined as:

$$\gamma_9=-\gamma_1\ldots\gamma_8\,, \qquad \gamma_9^2=\mathbb{1}\,. \tag{A.6}$$

The matrices $\gamma_A$ are analogously written by means of SO(7) gamma matrices $\mathbf{C}_A$:

$$\begin{aligned}\gamma_A &= \begin{pmatrix}0 & \mathbf{C}_A\\ \tilde{\mathbf{C}}_A & 0\end{pmatrix}\\ \tilde{\mathbf{C}}_A &= \mathbf{C}_A\ (A=1\ldots7)\,, \qquad -\mathbf{C}_A\ (A=8)\\ \mathbf{C}_8 &= \mathbb{1}\,,\end{aligned} \tag{A.7}$$

where $\mathbf{C}_A$ matrices are the octonion structure constants:

$$\begin{aligned}\mathbf{C}_A &= (\mathbf{C}_A)_{BC}\\ \mathbf{C}_{123} &= \mathbf{C}_{165}=\mathbf{C}_{257}=\mathbf{C}_{354}=\mathbf{C}_{367}=\mathbf{C}_{246}=\mathbf{C}_{147}=1\\ \mathbf{C}_{AB8} &= \delta_{AB}\end{aligned} \tag{A.8}$$

and antisymmetrization over the 3 indices is intended.

The matrix $\Gamma_{11}$ is:

$$\Gamma_{11}=-\Gamma_0\cdots\Gamma_9=\begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix} \tag{A.9}$$

moreover, due to our choice for the metric (A.2) we can say that:

$$\Gamma^+\Gamma^+=\Gamma^-\Gamma^-=0\,. \tag{A.10}$$

Matrices $W$ and $Y$ are:

$$W=\Gamma^1\Gamma^2\,, \qquad Y=\Gamma^3\Gamma^4 \tag{A.11}$$



JHEP07(2003)035

and because of their definition, satisfy some important properties:

$$\begin{aligned} W^2 &= Y^2 = -\mathbb{1}\,, \qquad [W,Y] = 0 \\ \left[W,\Gamma^{\pm}\right] &= \left[Y,\Gamma^{\pm}\right] = 0 \end{aligned} \tag{A.12}$$

notice that, due to the above properties, we are able to integrate the gravitino supersymmetry equation (4.5) and to write the solution in the form of equation (4.11). Because of Weyl condition ($\Gamma_{11}\Theta^I = -\Theta^I$, as the gravitino), $\Theta^I$ has only 16 nonzero components, and we can write:

$$\Theta^I = \begin{pmatrix} \theta^I \\ 0 \end{pmatrix}. \tag{A.13}$$

Due to light-cone gauge, we have an additional condition over $\Theta$:

$$\Gamma^+\Theta^I = 0 \to \left(\mathbb{1} + \gamma^9\right)\theta^I = 0 \tag{A.14}$$

this condition halves again the nonzero components of $\Theta^I$, but in our basis:

$$\gamma^9 = \begin{pmatrix} -\mathbb{1}_8 & 0 \\ 0 & \mathbb{1}_8 \end{pmatrix} \tag{A.15}$$

so, the only 8 components of $\Theta^I$ surviving are:

$$\Theta^I = \begin{pmatrix} S_8^I \\ 0_8 \\ 0_8 \\ 0_8 \end{pmatrix}. \tag{A.16}$$

## B. $AdS_n$ and $S_n$ spaces

The spaces of the type $AdS_n$ or $S_n$ can be viewed as hyper-surfaces in $\mathbb{R}^{n+1}$; in particular, let $z_\mu$ ($\mu = 1 \dots n+1$) be the coordinates of $\mathbb{R}^{n+1}$, then $AdS_n$ space is the hyperboloid:

$$z_0^2 - \sum_{i=1}^{n-1} z_i^2 + z_n^2 = -R^2 \tag{B.1}$$

defined in $\mathbb{R}^{n+1}$ with metric $(+,-,\cdots,-,+)$, while the $n$-sphere is:

$$\sum_{i=0}^{n} z_i^2 = R^2 \tag{B.2}$$

embedded in a space with metric $(-,\cdots,-)$. The coordinates of the embedding can be re-expressed in terms of the global ones:

$$AdS_n \to \begin{cases} z_0 = R\cosh\rho\cos t \\ z_n = R\cosh\rho\sin t \\ z_i = R\sinh\rho\;\Omega_i \end{cases} \tag{B.3}$$

$$S_n \to \begin{cases} z_0 = R\cos\theta\cos\psi \\ z_n = R\cos\theta\sin\psi \\ z_i = R\sin\theta\ \Omega_i\,, \end{cases} \tag{B.4}$$

where the only restriction on the form of $\Omega_i$ is that:

$$\sum_{i=1}^{n-1}\Omega_i^2 = 1\,. \tag{B.5}$$

Notice that $0 \leq \rho \leq +\infty$, while $-\pi \leq t, \psi, \theta \leq \pi$; in fact we need, in order to reinterpret $t$ as the time, to unwrap it and use the so-called global covering coordinates, where $-\infty \leq t \leq +\infty$.

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JHEP07(2003)035